Mathematics

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Important contacts

- **joint course by Maria Ekes and me**
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Grading

- 0-30 points: *average* of two written tests (mid-term and end-term, roughly 1/2 of the material)
- 0-30 points: final written exam (full material)
- 0-5 points: activity in exercise classes

score	grade
0-30	2
31-36	3
37-42	3.5
43-48	4
49-54	4.5
55+	5

Textbooks and other helpful resources

- official **e-book**
- WolframAlpha (desktop App or `https://www.wolframalpha.com/`
  - some **tutorials** are available
- Stewart James: Calculus Early Transcendentals, 2011, Brooks/Cole, Belmont CA,USA;
- Howard Anton, Chris Rorres: Elementary Linear Algebra with Suplemental Applications, 2010, Clarence Center Inc, Denver MA.

Key points

- definition of a sequence; arithmetic sequence; geometric sequence;
- bounded and monotone sequences;
- definition of a limit; simple arithmetic rules;
- squeeze theorem
- conditions for convergence;
- indeterminate forms

$$\frac{0}{0}, \frac{\infty}{\infty}, +\infty - \infty, 0 \cdot \infty, 1^{\infty}, 0^{0}, \infty^{0}$$

- the magical number **e**

Sequence

Definition (Sequence)

A sequence is a function $a : \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ is the set of natural numbers, and $\mathbb{R}$ is the set of real numbers. The value $a(n) = a_n$ is called the n-th term of the sequence.

Notation: a_n is a single number, while $(a_n)_{n=1}^{\infty}$ or $(a_n)_{n=1}^{\infty}$ or simply $(a_n)_n$ or $\{a_n\}_n$ denote a sequence.

Special sequences

- A sequence $(a_n)_n$ that satisfies: $a_{n+1} = a_n + r \quad \forall_{n>1}$ is called an **arithmetic sequence** with a common difference r.
- A sequence $(a_n)_n$ that satisfies: $a_{n+1} = a_n \cdot q \quad \forall_{n>1}$ is called a **geometric sequence** with a common ratio r.
- sequence is **(weakly) increasing** if $a_{n+1} \geq a_n \quad \forall_n$
- A sequence is **(weakly) decreasing** if $a_{n+1} \leq a_n \; \forall_n$
- A sequence is **bounded from below** if $\exists m \; \forall_n \; a_n \geq m$
- A sequence is **bounded from above** if $\exists M \; \forall_n \; a_n \leq M$
- A sequence is **bounded** if it is bounded from above and below

Limit of a sequence

Definition (Limit)

A number $g \in \mathbb{R}$ is a limit of a sequence $(a_n)_n$ if:

$$\forall_{\epsilon>0} \exists_N \forall_{n>N} \; |a_n - g| < \epsilon.$$

If such a number exists, we say the sequence ***converges to g.*** *Otherwise, the sequence is* ***divergent.***

Lemma

If $(a_n)_n$ has a limit, it is unique.

”Limits” in +∞ or ∞

Definition (Improper limit)

A sequence $(a_n)_n$ has an improper limit in $+\infty$ ($-\infty$) if:

$$\forall_M \exists_{N_M} \forall_{n>N_M}\ a_n > M \quad (a_N < M).$$

We say the sequence ***diverges to infinity (minus infinity)*** *and denote it by* $\lim_{n\to\infty} a_n = +\infty$ ($\lim_{n\to\infty} a_n = -\infty$) *or, shorter* $a_n \to \pm\infty$

Basic properties of finite limits

Assume $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ for $a, b, \in \mathbb{R}$:

- $\lim_{n\to\infty}(a_n \pm b_n) = (\lim_{n\to\infty} a_n) \pm (\lim_{n\to\infty} b_n) = a \pm b$
- $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ if only both sides well-defined i.e. $b_n, b \neq 0$
- $\lim_{n\to\infty} a_n^{b_n} = a^b$, if only both sides well-defined (note: 0^0 **not** well-defined)

Basic properties of limits with $\pm\infty$

Assume $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ and $\lim_{n\to\infty} b_n = +\infty$:

- $\lim_{n\to\infty}(a_n \pm b_n) = \pm\infty$
- $\lim_{n\to\infty}(a_n \cdot b_n) = \begin{cases} +\infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0 \\ \text{indeterminate} & \text{if } a = 0 \end{cases}$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$
- $\lim_{n\to\infty} a_n^{b_n} = \begin{cases} +\infty & \text{if } a > 1 \\ 0 & \text{if } |a| < 1 \\ \text{does not exist} & \text{if } a < -1 \\ \text{indeterminate} & \text{if } a = 1 \end{cases}$
- Note: rules for $b_n \to -\infty$ can be derived using last two slides;

When simple rules do not work

- when trying to apply arithmetic rules, we can calculate well-defined limits...
- ...but sometimes we encounter ill-defined statements:

$$\frac{0}{0}, \frac{\infty}{\infty}, +\infty - \infty, 0 \cdot \infty, 1^{\infty}, 0^0, \infty^0$$

- **indeterminate forms**
- important note: encountering indeterminate form does **not** necessarily mean the limit does not exist; it means we have to work harder to find it!
- important note 2: 'true' limits of statements in indeterminate form could be anything: a 'nice' number, zero, $-\infty$ etc.

”Classic” limits to remember

- exponential function diverges quicker than polynomial:

$$\lim_{n\to\infty} \frac{n^k}{2^n} = 0 \text{ for any } k > 0$$

(also works if we replace 2 with any number $a > 1$)

- factorial converges quicker than polynomial and exponential:

$$\lim_{n\to\infty} \frac{n^k}{n!} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{2^n}{n!} = 0$$

- for $a > 0$:

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} a^{\frac{1}{n}} = 1$$

- but also (which is less obvious):

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Squeeze theorem

Very, very useful

Theorem (Squeeze theorem)

Let sequences $(a_n)_n, (b_n)_n, (c_n)_n$, satisfy:

$$a_n \leq b_n \leq c_n \quad \forall_n \quad (\text{or at least } \exists_N \forall_{n>N})$$

Then if $a_n \to g$ and $c_n \to g$, it must be that $b_n \to g$.

Theorem (Divergence)

Let sequences $(a_n)_n, (b_n)_n$ satisfy:

$$a_n \leq b_n \quad \forall_n \quad (\text{or at least } \exists_N \forall_{n>N})$$

Then if $a_n \to \infty$, it must be that $b_n \to \infty$.

Euler's number

Theorem (Monotone convergence theoretm)

Every monotone and bounded sequence converges (to a proper limit).

Example

Let $a_n = \left(1 + \frac{1}{n}\right)^n$. We will show that the sequence a_n is strictly increasing and bounded and therefore has a limit.

Euler's number

Definition

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828$$

Note: this is a *definition* of e (one of few alternatives).

Lemma

Let $(a_n)_n$ be a sequence satisfying $a_n \to +\infty$ or $a_n \to -\infty$. Then:
$\left(1+\frac{1}{a_n}\right)^{a_n} = e$

Special case: $\left(1+\frac{c}{n}\right)^n = e^c$.

Function. Domain and image.

Definition

A function $f : x \to Y$ is relation that associates with each element of X exactly one element of Y.

- X is the **domain** of f (also denoted D_f, especially if we need to determine it!)
- Y is the **co-domain**; but usually we are interested in:
- $f(X) \subset Y$ is an **image** (sometimes: range) of f

Function one-to-one + ”onto” = bijection

- function is **one-to-one** (injective) if $x \neq y \Rightarrow f(x) \neq f(y)$;
  (it is usually easier to show equivalent statement $f(x) = f(y) \Rightarrow x = y$)
- function is **”onto”** (surjective) if $\forall_{y \in Y} \exists_{x \in X} : f(x) = y$
- function is **bijective** ($\Rightarrow$ invertible!) if it is one-to-one and ’onto’
- note: it hinges crucially on X and Y;
  take $f(x) = x^2$:
  - if $f : \mathbb{R} \to \mathbb{R}$ it is neither ’onto’ nor 1-1
  - if $f : \mathbb{R} \to [0, +\infty)$ it is ’onto’
  - if $f : [0, +\infty) \to \mathbb{R}$ it is ’onto’ and one-to-one
    and has an inverse! $f^{-1}(y) = \sqrt{y}$

Image and preimage of a set

Let $f : X \to Y$ be a function

- image of $A \subset X$ is:

$$\{f(x) \in Y : x \in A\}$$

  Notation: $f(A)$ or $f[A]$
- preimage of $B \subset Y$ is:

$$\{x \in X : f(x) \in B\}$$

  Notation: $f^{-1}(B)$ (not to be confused with an inverse function!) or $f^{-1}[B]$

Limit of a function

Definition (Limit points of a set)

*A point $x \in X$ is a **limit point** of X if there exists a seqence $(x_n)_n$ such that $x_n \in X \setminus x_0$ and $x_n \to x$. Otherwise, we call x an isolated point.*

Definition (Heine's limit)

Let x_0 be a limit point of X. A function $f(x) : X \to Y$ has a limit L in x_0 if for every sequence $(x_n)_n$ such that $x_n \in X \setminus x_0$ and $x_n \to x_0$ we have $f(x_n) \to L$.

Continuity

Definition

*A function f is **continuous** at $x_0 \in X$ if for every sequence $\lim_{n\to\infty} x_n \to x_0$ we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.*

- note: x_0 must be in domain of X to consider continuity!

Definition

*A function f is **continuous on the domain** (or: on a set $A \subset X$) if it is continuous in every point of its domain (or: of $A \subset X$).*

- sum, difference, product of continuous functions is continuous
- quotient of continuous functions is continuous *if well-defined* (do not divide by 0!)
- polynomial, exponential, rational, logarithmic,

Composition

Definition (Composition)

Let $f : X \to Y$ *and* $g : Y \to Z$. *Then a composition* $g \circ f =: h$ *if a function* $h : X \to Z$ *defined by:*

$$h(x) = g(f(x))$$

Lemma

A composition of continuous functions is a continuous function.

Asymptotes

Definition (Vertical asymptote)

A function $f(x)$ has a vertical asymptote $x = c$ if f has at least one-sided improper limit in c:

$$\lim_{x \to c^-} f(x) = \pm\infty \text{ or } \lim_{x \to c^+} f(x) = \pm\infty$$

- we do not require (and typically do not have) that $c \in D_f$.
- if f is continuous in x_0, there can't be an asymptote in x_0.
- $\Rightarrow$ vertical asymptotes may exist only in the limit points outside of the domain or in discontinuity points.

Asymptotes

Definition (Horizontal asymptote)

A function $f(x)$ has a horizontal asymptote $y = b$ if:

$$\lim_{x \to +\infty} f(x) = b \text{ or } \lim_{x \to -\infty} f(x) = b$$

Definition (Oblique asymptote)

A function $f(x)$ has an oblique asymptote $y = ax = b$ if:

$$\lim_{x \to +\infty} (f(x) - ax - b) = 0 \text{ or } \lim_{x \to -\infty} (f(x) - ax - b) = 0$$

Useful rules for oblique asymptotes:

$$a := \lim \frac{f(x)}{x}, \quad b := \lim f(x) - ax$$

Three important limits

It is useful to remember those three rules:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Click here for a beaufiful visual proof.

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

Here is a proof with e, but I'll also show another one.

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

Another visual proof.

Inverse function

- A function is bijective (1-1 and onto) *if and only if* it has an inverse.
- Inverse function f^{-1}:

$$f^{-1}(y) = x \;\Leftrightarrow\; f(x) = y$$

- Alternative definition/property:

$$f^{-1}(f(x)) = x \;\text{ and }\; f(f^{-1}(x)) = x$$

Example - inverse trigonometric functions

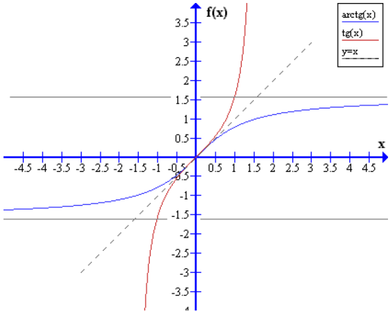

Figure: arctangent
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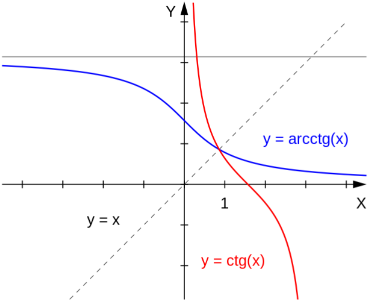

Figure: arccotangent
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Derivative

Definition (Derivative)

Let $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$. If there exists a finite limit

$$\lim_{h \to 0} \frac{f(x_0 + h - f(x_0)}{h},$$

we call it the derivative of f in x_0 & denote it by $f'(x_0)$ or $\frac{\partial f}{\partial x}(x_0)$. Function f' defined in all points x in which $f'(x)$ exists shall be called the (first) derivative of f.

- $\frac{f(x_0+h-f(x_0)}{h}$ or $\frac{f(x_1)-f(x_0)}{x_1-x_0}$ is the **difference quotient**
- if f' exists (on some set), we call it **differentiable** (on this set)
- if function is differentiable on (a, b), it is continuous on (a, b)

One-sided derivatives

Definition (Left (right) derivative)

A left derivative is a limit

$$f'_{-}(x_0) = \lim_{h \to 0^-} \frac{f(x_0 + h - f(x_0)}{h}$$

$$\left(f'_{+}(x_0) = \lim_{h \to 0^+} \frac{f(x_0 + h - f(x_0)}{h} \right)$$

A function is differentiable at x_0 if and only if its left and right derivatives exist and are equal.

Example

Function $f(x) = |x|$ is nondifferentiable in $x = 0$.

Derivatives of basic funtions

- $(x^a)' = ax^{a-1}$
- in particular: $(c)' = 0$
- $(e^x)' = e^x$
- $(a^x)' = a^x \cdot \ln(a)$
- $(\ln(x))' = \frac{1}{x}$
- $(\sin(x))' = \cos(x)$
- $(\cos(x))' = -\sin(x)$

Rules of differentiation

- $(c \cdot f(x))' = c \cdot f'(x)$
- $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
- $(g(f(x)))' = g'(f(x)) \cdot f'(x)$

The last one is **very important!**

Tangent line

Definition (Tangent)

Let $f : (a, b) \Rightarrow \mathbb{R}$ be a function differentiable at x_0 then a line:

$$y = f'(x_0)(x - x_0) + f(x_0)$$

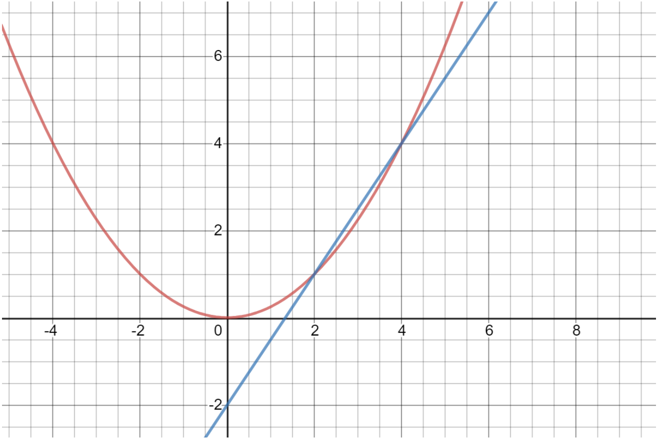
*is **tangent to the curve** at x_0.*

Note that the definition follows easily from the definition of the derivative:

$$f'(x_0) \approx \frac{f(x) - f(x_0)}{x - x_0}$$

The value $f'(x_0)$ is the slope/gradient of line, i.e. $f'(x_0) = a = \alpha$, where α is the angle of incline.

Tangent as a limit of secant lines



Click to go to interactive graph

Derivatives in economics

Derivative = change of the function value after a ’small’ unit change in its argument.

$$f'(x_0) \approx \frac{f(x) - f(x_0)}{x - x_0}$$

- marginal cost/revenue = derivative of the total cost/revenue function
- optimization problems (e.g. maximizing utility)
- elasticities (see next slide)

Elasticities

Derivative = change of the function value after a ’small’ unit change in its argument.
Elasticity = **percentage** change of the function value after a ’small’ **percentage** change in its argument.

$$E_f(x_0) = \frac{f'(x_0)x_0}{f(x_0)}$$

- price elasticity of demand
- price elasticity of supply
- cross elasticities